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The positive mass theorem for black holes revisited

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Abstract

We present a rigorous proof of the positive mass theorem for black holes. Accordingly, in a four-dimensional Lorentz manifold satisfying the dominant energy condition, the mass of a three-dimensional asymptotically flat slice with boundary composed of a finite number of future or past trapped closed 2-surfaces is nonnegative. The proof uses the classical Witten argument and is valid even if only rather weak asymptotic conditions are imposed.

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0. Introduction

Soon after the appearance of Witten's [12] proof of the positive mass theorem using spinors, Gibbons et al. [4] (and independently Reula and Tod [11] in the asymptotically hyperbolic case) proposed to extend this proof to asymptotically flat manifolds containing one or several black holes, since physical as well as mathematical intuition suggested that the mass should also be nonnegative in this setting. The proof proposed in [4] demanded that the manifold should admit a nice conformal compactification at infinity and some arguments were missing. It has now appeared [3] that it might be useful to have a rigorous proof of such an extension with asymptotic conditions that are considerably weaker than the ones considered in [4]. This paper aims at filling this gap.

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We shall work on a four-dimensional Lorentz manifold (N, γ) with signature $(-+++)$. M will be a Riemannian hypersurface whose induced metric will be denoted by g. We moreover assume that it is asymptotically flat, in the sense that in some chart at infinity

$$g - e \in C^{2,\alpha}_{\varepsilon}, \qquad h \in C^{1,\alpha}_{\varepsilon+1},$$

where e is the Euclidean metric of the chart, h the second fundamental form of M in N and the $C^{k,\alpha}_{\varepsilon}$'s are weighted Hölder spaces defined below. M being asymptotically flat, if $\varepsilon > \frac{1}{2}$ and if the scalar contraint $\mathrm{Scal}^g - |h|^2_g + (\mathrm{tr}_g\, h)^2$ (where Scal^g is the scalar curvature of g) is in L^1, its mass is defined [1] as

$$m = \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r} (\partial_i g_{ij} - \partial_j g_{ii}) \nu^j_r \,\mathrm{d}\,\mathrm{vol}_{S_r},$$

where S_r is the sphere of radius r in a chart at infinity and ν_r its outward unit normal. Notice that our asymptotic conditions only imply that the metric should decay faster than $r^{-1/2}$ at infinity, a decay which may not be compatible with a conformal compactification procedure. If moreover the divergence constraint $\delta^g h - d(\mathrm{tr}_g\, h)$ is also in L^1, its momentum with respect to an asymptotically constant vector X is defined by

$$\mu(X) = \frac{1}{8\pi} \lim_{r\to\infty} \int_{S_r} (h(\nu_r, X) - g(X, \nu_r)\mathrm{tr}\, h)\,\mathrm{d}\,\mathrm{vol}_{S_r}.$$

Mass and momentum are usually seen as the components of a 4-vector, the energy–momentum vector $\mu = (\mu^0 = m, \mu^1, \mu^2, \mu^3)$. We shall then prove:

Positive mass theorem (for black holes). *Suppose (N, γ) satisfies the dominant energy condition, and that M is an asymptotically flat Riemannian hypersurface such that its energy–momentum is defined and its boundary is composed of a finite number of either past or future trapped closed 2-surfaces in N. Then*

$$|\mu|^2 = m^2 - \sum_{i=1}^{3} (\mu^i)^2 \geq 0.$$

1. Geometric preliminaries

As already mentioned, we shall work with (4-component) spinors. Recall that N is endowed with an $SL(2,\mathbb{C})$-principal bundle P_Σ of spinorial frames and that the choice of a unit normal e_0 of M in N gives an embedding of the $SU(2)$-principal bundle of spinorial frames of M into P_Σ. The usual (4-component) spinor bundle on N is

$$\Sigma = P_\Sigma \times_{\rho\oplus\bar{\rho}} \mathbb{C}^2 \oplus \overline{\mathbb{C}^2},$$

where ρ is the usual representation of $SL(2,\mathbb{C})$ on $\mathbb{C}^2$. The spinor bundle Σ is then endowed with two inner products, the first one, coming from $SL(2,\mathbb{C})$-invariance, will be denoted by $(\cdot,\cdot)$; the other one, coming from $SU(2)$-invariance, is Hermitian positive definite and is defined by $\langle\cdot,\cdot\rangle = (e_0\cdot\cdot,\cdot)$ where $\cdot$ stands for Clifford multiplication of vectors upon spinors. Clifford multiplication by any vector is Hermitian with respect to $(\cdot,\cdot)$, Clifford multiplication by any vector tangent to M is anti-Hermitian for $\langle\cdot,\cdot\rangle$ whereas Clifford multiplication by e_0 is Hermitian. The bundle Σ is also endowed with two different connections, coming from the four-dimensional structure (and denoted by ∇) and the three-dimensional one (denoted by $\overline{\nabla}$). They are related by the formula

$$\nabla_X\psi = \overline{\nabla}_X\psi - \tfrac{1}{2}h(X)\cdot e_0\cdot\psi,$$

where h is the symmetric endomorphism associated to the second fundamental form (note that our conventions in this article are that $h_{ij} = g(\nabla_{e_i}e_j, e_0)$). On the contrary of Clifford multiplication, both covariant derivatives respect the decomposition of Σ into half-spinor (2-component) representations. The Dirac–Witten operator is then defined on a spinor ψ at a point x in M as

$$\mathcal{D}\psi = \sum_{i=1}^{3} e_i\cdot\nabla_{e_i}\psi,$$

where the summation is over any orthonormal basis of vectors of T_xM.

We shall consider here a Riemannian slice M that has an inner boundary ∂M. If $\nabla\!\!\!\!/$ is the Levi-Civita connection of its induced metric, ν its outer unit normal and θ its second fundamental form (with the same convention as above, so that a round sphere in the flat Euclidean space would have here negative second fundamental form with respect to its outer normal), the following formula holds on spinors:

$$\overline{\nabla}_X\psi = \nabla\!\!\!\!/_X\psi + \tfrac{1}{2}\theta(X)\cdot\nu\cdot\psi.$$

Any component of the boundary is a future (resp. past) trapped (or future/past converging) surface if its mean curvature vector is causal and past (resp. future) pointing [9, Definition 14.57, p. 435]. In our conventions, this means that on the whole of it,

$$\operatorname{tr} h - h(\nu,\nu) \geq |\operatorname{tr}\theta| \qquad (\text{resp. } -\operatorname{tr} h + h(\nu,\nu) \geq |\operatorname{tr}\theta|).$$

We will denote by M_r the domain contained between the boundary and a large geodesic sphere in M, denoted itself by S_r.

We also need to define the weighted Hölder spaces (of spinors or tensors)

$$C^{k,\alpha}_\beta = \{u \in C^{k,\alpha}_{\mathrm{loc}},\ \|r^\beta u\|_{C^0} < \infty, \ldots, \|r^{\beta+k}D^k u\|_{C^0} < \infty, r^{k+\beta+\alpha}[D^k u]_\alpha < \infty\}$$

where

$$[D^k u]_\alpha = \sup_{|z-z'|\leq 1} |z-z'|^{-\alpha}|D^k u(z) - D^k u(z')|,$$

and the $\{z_i\}$ are the coordinates of any chart at infinity (with $r = |z|$), and the weighted Sobolev spaces, defined as in [10]:

$$H^k_\beta = \{u \in H^k_{\mathrm{loc}},\ r^{\beta+l}u \in L^2\ \forall l \le k\},$$
$$W^{k,p}_\beta = \{u \in W^{k,p}_{\mathrm{loc}},\ r^{\beta+l}u \in L^q\ \forall l \le k\}.$$

The proof of the positive mass theorem relies on two different tools: an integration by parts formula or Bochner–Lichnerowicz–Weitzenbock formula for the Dirac–Witten operator and an existence theorem for some asymptotically constant spinor sitting in the kernel of $\mathcal{D}$. We shall now recall the formula which is by now well-known and we defer the analysis to the next section.

Lemma 1. *The Bochner–Lichnerowicz–Weitzenbock–Witten formula states*

$$\mathcal{D}^*\mathcal{D} = \mathcal{D}^2 = \nabla^*\nabla + \mathcal{R},$$

where $\mathcal{R}$ is the endomorphism of the spinor bundle defined on a spinor φ by

$$\begin{aligned}\mathcal{R}\varphi &= \tfrac{1}{4}(\mathrm{Scal}^\gamma + 4\,\mathrm{Ric}^\gamma(e_0, e_0) + 2e_0 \cdot \mathrm{Ric}^\gamma(e_0)\cdot)\varphi \\ &= \frac{1}{2}\left(G(e_0, e_0) + \sum_{i=1}^{3} G(e_0, e_i)e_0 \cdot e_i\cdot\right)\varphi,\end{aligned}$$

where G is the Einstein tensor $G = \mathrm{Ric}^\gamma - \frac{1}{2}\mathrm{Scal}^\gamma\gamma$. If the dominant energy condition is satisfied in N, then $\mathcal{R}$ is nonnegative.

Lemma 2. *For any smooth spinor field ψ, if ν_r the outer unit normal of S_r,*

$$\begin{aligned}\int_{M_r}\langle\mathcal{D}\psi, \mathcal{D}\psi\rangle &= \int_{M_r}\langle\nabla\psi, \nabla\psi\rangle + \int_{M_r}\langle\psi, \mathcal{R}\psi\rangle + \int_{\partial M}\langle\nabla_\nu\psi + \nu\cdot\mathcal{D}\psi, \psi\rangle \\ &\quad - \int_{S_r}\langle\nabla_{\nu_r}\psi + \nu_r\cdot\mathcal{D}\psi, \psi\rangle.\end{aligned}$$

2. Analysis of the Dirac–Witten operator

We now enter the main technical part of the paper. We are going to prove that the Dirac–Witten operator, together with some well-chosen boundary conditions defines an isomorphism between some adapted weighted Sobolev spaces.

Let us first define on the restriction of the bundle Σ over ∂M the endomorphism

$$\varepsilon\varphi := \nu \cdot e_0 \cdot \varphi.$$

Lemma 3. *ε is symmetric, it anticommutes to the Clifford action of the normals ν or e_0 and it commutes to the action of any vector tangent to the boundary. It also respects the decomposition into 2-component spinors.*

Since $\varepsilon^2 = 1$, the spinor bundle restricted over the boundary splits into the $+1$ and -1 eigenbundles of ε, denoted by Σ_+ and Σ_- (be careful this has nothing to do with the decomposition of the spinor bundle into half-spinor representations).

Let us now define two operators

$$\mathfrak{L}_{1,\pm} : H^1_{-1}(M, \Sigma) \longrightarrow L^2(M, \Sigma) \times H^{1/2}(\partial M, \Sigma_\pm), \quad \psi \longmapsto (\mathcal{D}\psi, \varepsilon\psi \pm \psi)$$

and

$$\mathfrak{L}_{2,\pm} : H^2_{-1}(M, \Sigma) \longrightarrow L^2_1(M, \Sigma) \times H^{3/2}(\partial M, \Sigma_\pm) \times H^{1/2}(\partial M, \Sigma_\pm),$$
$$\psi \longmapsto (\mathcal{D}^2\psi, \varepsilon\psi \pm \psi, \varepsilon(\mathcal{D}\psi) \pm \mathcal{D}\psi).$$

Our main goal in this section is to show that $\mathfrak{L}_{2,\pm}$ is an isomorphism in the chosen weighted spaces.

Following Bunke's results [2], we first show that the boundary conditions

$$\varepsilon\psi \pm \psi, \qquad \varepsilon(\mathcal{D}\psi) \pm \mathcal{D}\psi$$

both satisfy the Lopatinski–Shapiro condition for the operators $\mathfrak{L}_{2,\pm}$. Let us do it for the "$-$" sign: we want, for any vector ξ, tangent to the boundary,

$$u \quad \longmapsto \quad (\varepsilon u - u, \varepsilon((\mathrm{i}\xi + \mathrm{i}D_t\nu) \cdot u) - (\mathrm{i}\xi + \mathrm{i}D_t\nu) \cdot u)_{|t=0}$$

to be a bijection from the space of bounded solutions of the ODE (complex dimension 4)

$$|\xi + \nu D_t|^2 u(t) = 0$$

($D_t = -\mathrm{i}\partial_t$ as usual) to the spinor bundle restricted to the boundary (same dimension). Consider injectivity: solutions of the ODE are of the form

$$u(t) = u_0\, \mathrm{e}^{-|\xi|t}, \quad u_0 \in \Sigma$$

and being in the kernel of the boundary operator implies

$$\varepsilon u_0 = u_0, \qquad \varepsilon(\mathrm{i}\xi \cdot u_0 - |\xi|\nu \cdot u_0) = \mathrm{i}\xi \cdot u_0 - |\xi|\nu \cdot u_0,$$

so that

$$(u_0)^+_- = 0, \qquad (u_0)^-_- = 0$$

and

$$[\mathrm{i}\xi \cdot u_0 - |\xi|\nu \cdot u_0]^+_- = 0, \qquad [\mathrm{i}\xi \cdot u_0 - |\xi|\nu \cdot u_0]^-_- = 0,$$

where the exponents $\pm$ refer to the half-spin representations and the indices $\pm$ to the corresponding eigensub-bundles relative to ε. The commutation properties of the vectors with ε then imply the desired result.

We shall now consider here the case where the boundary is only one past trapped surface (this means that $\pm$ will have the value $+$ in the definition of the operators $\mathfrak{L}_{i,\pm}$ above). The general case can be easily obtained along the same lines. For the sake of simplicity, the operators $\mathfrak{L}_{i,+}$ will simply be denoted by $\mathfrak{L}_i$.

Proposition 4. *Suppose that* $\operatorname{tr}\theta - \operatorname{tr} h + h(\nu, \nu) \geq 0$ *along the boundary. Then* $\mathfrak{L}_2$ *is injective from* H^2_{-1} *to* $L^2_1 \times H^{3/2} \times H^{1/2}$.

Proof. The proof relies on the fact that $\mathfrak{L}_1$ is itself injective. We first compute a general formula. If $\psi \in C^\infty_c$, then

$$\lim_{r\to\infty} \int_{S_r} \langle \nabla_{\nu_r} + \nu_r \cdot \mathcal{D}\psi, \psi \rangle = 0.$$

From Lemma 2, we get that

$$\int_M \langle \mathcal{D}\psi, \mathcal{D}\psi \rangle = \int_M \langle \nabla\psi, \nabla\psi \rangle + \int_M \langle \psi, \mathcal{R}\psi \rangle + \int_{\partial M} \langle \psi, \nabla_\nu \psi + \nu \cdot \mathcal{D}\psi \rangle.$$

We now compute the surface term, which is the only nonpositive term.

Lemma 5. *For any smooth* ψ,

$$\nabla_\nu \psi + \nu \cdot \mathcal{D}\psi = \sum_{k=2}^{3} \left(\nu \cdot e_k \cdot \overline{\nabla}_{e_k} \psi - \frac{1}{2} h(e_k, \nu) e_k \cdot e_0 \cdot \psi \right) + \tfrac{1}{2} \left(\operatorname{tr}\theta - (h(\nu, \nu) - \operatorname{tr} h)\varepsilon\right) \psi,$$

where e_2, e_3 *is an orthonormal basis of* $T\partial M$ *and* $\overline{\nabla}$ *is the Levi-Civita connection of* ∂M.

Proof of Lemma 5. From the expression of the Dirac–Witten operator, we get

$$\nu \cdot \mathcal{D}\psi + \nabla_\nu \psi = \sum_{k=2,3} \nu \cdot e_k \cdot \nabla_{e_k} \psi$$

and the formula follows by expressing the four-dimensional and three-dimensional connection coefficients with respect to the three-dimensional and two-dimensional connection coefficients together with their second fundamental forms. □

Proof of Proposition 4 (continued). From the hypotheses on ε, we infer that both $\nu \cdot e_k \cdot$ and $e_k \cdot e_0 \cdot$ exchange its eigenbundles, so that the term where they appear does not contribute to $\langle \psi, \nabla_\nu \psi + \nu \cdot \mathcal{D}\psi \rangle$ (notice that $\varepsilon \cdot$ and $\nu \cdot$ are not ∇-parallel, whereas they are $\overline{\nabla}$-parallel; this is the reason why we need to express anything with respect to the 2-dimensional covariant derivative). The hypothesis on the dominant energy condition then imply that

$$\int_M \langle \mathcal{D}\psi, \mathcal{D}\psi \rangle \geq \int_M \langle \nabla\psi, \nabla\psi \rangle + \frac{1}{2} \int_{\partial M} \langle \psi, (\operatorname{tr}\theta - (h(\nu, \nu) - \operatorname{tr} h)\varepsilon)\psi \rangle.$$

Since C^∞_c is dense in H^1_{-1}, the formula is valid for any ψ in H^1_{-1}.

Hence, for any spinor field in the kernel of $\mathfrak{P}_1$,

$$0 = \int_M \langle \mathcal{D}\psi, \mathcal{D}\psi \rangle \ \geq \ \int_M \langle \nabla\psi, \nabla\psi \rangle$$

and, since the right-hand side is a Hilbert norm on H^1_{-1}, the spinor is identically zero.

Consider now a spinor in the kernel of $\mathfrak{P}_2$. Since H^1_0 is included in H^1_{-1}, injectivity of $\mathfrak{P}_1$ applied twice gives injectivity of $\mathfrak{P}_2$. □

We now proceed to the proof of surjectivity.

Proposition 6. *Under the hypothesis* $\operatorname{tr}\theta - \operatorname{tr} h + h(\nu, \nu) \geq 0$, *the operator* $\mathfrak{P}_2$ *is also surjective.*

Proof. We are going to find a solution of

$$\mathfrak{P}_2\psi = (\varphi, \varphi_0, \varphi_1) \in L^2_1 \times H^{3/2} \times H^{1/2}.$$

Choose $\underline{\psi}$ in H^2_{-1} such that

$$(\varepsilon\underline{\psi} + \underline{\psi}, \varepsilon(\mathcal{D}\underline{\psi}) + \mathcal{D}\underline{\psi}) = (\varphi_0, \varphi_1).$$

Since the bilinear form

$$\int_M \langle \mathcal{D}\psi, \mathcal{D}\psi \rangle$$

is coercive on $\{\xi \in H^1_{-1}\ ,\ \varepsilon\xi + \xi = 0\}$, the Lax-Milgram lemma then provides, for any φ in L^2_1, a unique ψ in $\{\xi \in H^1_{-1}\ ,\ \varepsilon\xi + \xi = 0\}$ such that

$$\int_M \langle \mathcal{D}\psi, \mathcal{D}\xi \rangle = \int_M \langle \eta, \xi \rangle \quad \forall \xi \in \{\xi \in H^1_{-1}, \varepsilon\xi + \xi = 0\},$$

where $\eta = \varphi - \mathcal{D}^2\underline{\psi}$. If the spinor field was smooth enough, we would then get by integration by parts

$$\begin{aligned} 0 &= \int_M \langle \mathcal{D}\psi, \mathcal{D}\xi \rangle - \langle \varphi - \mathcal{D}^2\underline{\psi}, \xi \rangle \\ &= \int_M \langle \mathcal{D}^2(\psi + \underline{\psi}) - \varphi, \xi \rangle + \int_{\partial M} \langle \mathcal{D}\psi, \nu \cdot \xi \rangle. \end{aligned}$$

We would then have, in a weak sense,

$$\mathcal{D}^2\psi + \mathcal{D}^2\underline{\psi} - \varphi = 0, \quad \varepsilon\psi + \psi = 0, \quad \varepsilon(\mathcal{D}\psi) + \mathcal{D}\psi = 0.$$

But the boundary conditions given here satisfy the Lopatinski–Shapiro condition of ellipticity; if our operators have smooth coefficients, we can apply the results of classical pseudo-differential calculus [7, Chap. XX] and we conclude that ψ has local regularity H^2 (including around the boundary) and the last PDE system is valid in the strong sense.

If the local regularity of the metric is only $C^{2,\alpha}$, we shall prove this regularity result using the classical translations (or difference-quotient) method due to Nirenberg: this is quite classical but the author has been unable to find a reference where is stated a theorem applying directly to our situation (the closest being likely the work of Morrey on differential forms [8, Chap. 7]):

Lemma 7. *The spinor field ψ is in H^2 up to the boundary.*

Proof. We consider a local trivialization of the spinor bundle in a tubular neighborhood of a small open set $\mathcal{U}$ in the boundary:

$$\Sigma M_{|\mathcal{U}\times[0,\varepsilon[} = \mathcal{U}\times[0,\varepsilon[\times\mathbb{C}^4,$$

where the fiber $\mathbb{C}^4$ is further decomposed into $\mathbb{C}^2\oplus\mathbb{C}^2$, where the factors are the eigenspaces of ε on ∂M with respect to the eigenvalues ± 1. We denote by R_h a 1-parameter family of diffeomorphisms of $\mathcal{U}$ (take for example the flow of some vector field Y of compact support in $\mathcal{U}$), extended to $\mathcal{U}\times[0,\varepsilon[$, and we define, for any $\xi\in\{\xi\in H^1_{-1},\ \varepsilon\xi+\xi=0\}$,

$$R_h\xi = (x,(R_h\xi)(x)) = (x,\xi(R_hx))$$

in the previous trivialization. This provides us with a nice approximation of the derivative

$$D_h\xi = \frac{1}{h}\left(R_h\xi-\xi\right),$$

in the direction of Y, which respects the space $\{\xi\in H^1_{-1},\ \varepsilon\xi+\xi=0\}$.

It is classical that the second derivative $\nabla\nabla_Y\xi$ lives in L^2 if

$$\|\ \nabla D_h\xi\ \|_{L^2(\mathcal{U}\times[0,\varepsilon[)}$$

is bounded uniformly in h. We shall now prove this property for $\nabla\nabla_Y\xi$ with Y tangent to the boundary.

Consider the spinor field ψ previously found. Using a cut-off function we can restrict ourselves to a spinor field (still denoted by ψ) supported in a neighborhood of $\mathcal{U}\times[0,\varepsilon[$, which satisfies

$$\forall\xi\in\{\xi\in H^1_{-1},\mathrm{Supp}(\xi)\subset\mathcal{U}\times[0,\varepsilon[,\varepsilon\xi+\xi=0\},$$
$$\int_{\mathcal{U}\times[0,\varepsilon[}\langle\mathcal{D}\psi,\mathcal{D}\xi\rangle=\int_{\mathcal{U}\times[0,\varepsilon[}\langle\eta,\xi\rangle,$$

where η is an L^2 spinor field. Let $\xi=D_{-h}D_h\psi$. Then

$$\int_{\mathcal{U}\times[0,\varepsilon[}\langle\mathcal{D}\psi,\mathcal{D}(D_{-h}D_h\psi)\rangle=\int_{\mathcal{U}\times[0,\varepsilon[}\langle\eta,D_{-h}D_h\psi\rangle,$$

so that after a change of variables $y = R_h x$ in the LHS,

$$\int\limits_{\mathcal{U}\times[0,\varepsilon[} \langle \mathcal{D}(D_h\psi), \mathcal{D}(D_h\psi)\rangle = \int\limits_{\mathcal{U}\times[0,\varepsilon[} \langle \eta, D_{-h}D_h\psi\rangle + \text{correction terms}$$

where the "correction terms" include terms like

$$\int\limits_{\mathcal{U}\times[0,\varepsilon[} \langle [\mathcal{D}, D_{-h}]\psi, \mathcal{D}D_h\psi\rangle \quad \text{or} \quad \int\limits_{\mathcal{U}\times[0,\varepsilon[} \langle \mathcal{D}\psi, [\mathcal{D}, D_{-h}]D_h\psi\rangle.$$

From this we conclude

$$\int\limits_{\mathcal{U}\times[0,\varepsilon[} \langle \mathcal{D}(D_h\psi), \mathcal{D}(D_h\psi)\rangle \leq \|\eta\|_{L^2} \parallel D_{-h}D_h\psi \parallel_{L^2} + C_1 \parallel \nabla D_h\psi \parallel_{L^2},$$

where C_1 is a constant containing the H^1-norm of ψ, the operator norm of the commutators $[\mathcal{D}, R_h]$ and $[\mathcal{D}, R_{-h}]$ (themselves estimated from the weighted C^2-norm of the metric), and the weighted C^1-norm of the metric again. Denoting

$$\sigma_h = \|\nabla D_h\psi\|_{L^2(\mathcal{U}\times[0,\varepsilon[)},$$

the coercivity estimate eventually gives

$$(\sigma_h)^2 \leq C_1\,\sigma_h,$$

from which the conclusion follows.

We then get, from the definition of Sobolev spaces H^2, that there exists a constant C_2 such that for any pair of unit directions Y and Z (such that one at least is tangent to the boundary) and any smooth compactly supported ξ,

$$\int\limits_{\mathcal{U}\times[0,\varepsilon[} \langle \nabla_Y\psi, \nabla_Z\xi\rangle \leq C_2\|\xi\|_{L^2},$$

and this provides the correct estimates with one derivative (at least) tangent to the boundary.

Moreover,

$$\begin{aligned}
\langle \mathcal{D}\psi, \mathcal{D}\xi\rangle &= \left\langle \nabla_\nu\psi - \nu\cdot\sum_{i=1}^{2} e_i\cdot\nabla_{e_i}\psi, \nabla_\nu\xi - \nu\cdot\sum_{i=1}^{2} e_i\cdot\nabla_{e_i}\xi \right\rangle \\
&= \langle \nabla_\nu\psi, \nabla_\nu\xi\rangle + \left\langle \nu\cdot\sum_{i=1}^{2} e_i\cdot\nabla_{e_i}\psi, \nu\cdot\sum_{i=1}^{2} e_i\cdot\nabla_{e_i}\xi \right\rangle \\
&\quad + \left\langle \nu\cdot\nabla_\nu\psi, \sum_{i=1}^{2} e_i\cdot\nabla_{e_i}\xi \right\rangle + \left\langle \sum_{i=1}^{2} e_i\cdot\nabla_{e_i}\psi, \nu\cdot\nabla_\nu\xi \right\rangle
\end{aligned}$$

so that from the formula

$$\int\limits_M \langle \mathcal{D}\psi, \mathcal{D}\xi\rangle = \int\limits_M \langle \eta, \xi\rangle$$

valid for any smooth compactly supported ξ and the previous estimates for derivatives tangent to the boundary, we get that

$$\int_M \langle \nabla_\nu \psi, \nabla_\nu \xi \rangle \leq 8C_2 \|\xi\|_{L^2} + \|\eta\|_{L^2} \|\xi\|_{L^2},$$

which shows that the normal derivative $\nabla_\nu \nabla_\nu \psi$ also lives in L^2. This ends the proof of the lemma. □

We now return to our solution ψ. Let us now define $\overline{\psi} = \psi + \underline{\psi}$. From ellipticity of the Dirac operator, we also get that for any spinor field ξ living in some H^2_β,

$$\|\xi\|_{H^2_\beta(M\setminus K_1)} \leq C(\|\mathcal{D}^2\xi\|_{L^2_{\beta+2}(M\setminus K_2)} + \|\xi\|_{L^2_\beta(M\setminus K_2)}),$$

where $K_2 \subset K_1$ are compact subsets of M containing the boundary. This inequality is indeed obtained by patching together the classical local inequalities [5] using Bartnik's scaling argument (see [1, Proposition 1.15]).

Applying to $\beta_R \overline{\psi}$ (where β_R is a cut-off function which is zero outside a ball of radius R and satisfies $|\mathrm{d}\beta_R| \leq c\, R^{-1}$, $|D\, \mathrm{d}\beta_R| \leq c\, R^{-2}$) and letting R tend to infinity shows that $\overline{\psi}$ belongs to H^2_{-1} and is the desired solution. $\mathfrak{L}_2$ is then surjective. □

Remarks.

1. Since H^2_{-1} is included in $C^{0,\alpha}_{1/2}$, the same kind of estimate (in the Hölder classes) shows further that our spinor field is contained in $C^{2,\alpha}_{1/2}$ (then in $C^{2,\alpha}_{\varepsilon'}$ with some $\varepsilon' > \frac{1}{2}$ since there are no critical weights of the Dirac operator between 0 and 1) around infinity.
2. The isomorphism result is also true if the metric g (resp. second fundamental form h) lives in the Sobolev spaces $W^{2,q}_{\varepsilon-3/q}$ (resp. $W^{1,q}_{\varepsilon+1-3/q}$) with $\varepsilon \geq \frac{1}{2}$, $q > 3$. This is proved in Appendix A.
3. If the past trapped condition is replaced by a future trapped one, the same conclusions will hold with the + sign replaced by a − in the definition of the operators. If the boundary is a finite disjoint union of future or past trapped surfaces, we need to mix both kinds of boundary conditions.
4. Notice also that the positivity of boundary term is obtained independently of the fact that $\mathcal{D}\psi = 0$. Thus, the idea of *gauge choice* founding [10] may be still applicable in our case.

Proof of the positive mass theorem. It is done as usual: let ψ_0 be a spinor field which is constant in some chart around infinity, smooth and identically zero around the boundary. Then the conditions on the decay at infinity of the metric imply that $\nabla\psi_0$ lives in H^1_0. We can then find $\psi = \psi_0 + \psi_1$ with ψ_1 in H^2_{-1}, such that $\mathfrak{L}_2\psi = 0$. Injectivity of $\mathfrak{L}_1$ then shows that

$$\mathcal{D}\psi = 0, \qquad \varepsilon\psi + \psi = 0.$$

But Lemma 2 together with the fact that

$$4\pi\left\langle \psi_0, m\psi_0 - \sum_{1\le i\le 3} \mu^i\, \mathrm{d}x^0 \cdot \mathrm{d}x^i \cdot \psi_0 \right\rangle = \lim_{r\to\infty} \int_{S_r} \langle \nabla_{\nu_r}\psi + \nu_r \cdot \mathcal{D}\psi, \psi\rangle$$

(where the x^α's stand for the Euclidean coordinates of the chosen chart at infinity) give the positive mass theorem. □

Remark. As in [4], the positive mass theorem can be extended to charged black holes, thus implying that the mass must be bigger than the charge. The proof goes exactly along the same lines with the Levi-Civita connection ∇ replaced by

$$\nabla^{\omega}_X\psi = \nabla_X\psi - \frac{\mathrm{i}}{2}\,\omega(e_0,\cdot)\cdot e_0 \cdot X \cdot \psi + \frac{\mathrm{i}}{2} \sum_{1\le a<b\le 3} \omega(e_a, e_b) e_a \cdot e_b \cdot X \cdot \psi,$$

where ω is the electromagnetic field. If the dominant energy condition (see [4,6]) is satisfied, the Einstein and Maxwell equations imply that the zeroth-order term in the Weitzenböck formula is still nonnegative, and the boundary terms have the right commutation properties with respect to ε. As in the previous proof, this implies that we can find an asymptotically constant spinor, harmonic for the twisted Dirac–Witten operator

$$\mathcal{D}^{\omega} = \sum_{i=1}^{3} e_i \cdot \nabla^{\omega}_{e_i}.$$

Moreover, the boundary term now gives the inequality

$$m^2 - \sum_{1\le i\le 3} (\mu^i)^2 \ge P^2 + Q^2,$$

where P and Q are the magnetic and electric charges of the black hole.

Appendix A

We shall prove here that the operator $\mathfrak{L}_2$ is still an isomorphism if the metric is not $C^{2,\alpha}$ but has only local regularity in Sobolev spaces, namely when $g_{ij} - \delta_{ij}$ is in $W^{2,q}_{\varepsilon-3/q}$, the Einstein tensor is in $L^1 \cap L^\infty$ and the second fundamental form is in $W^{1,q}_{\varepsilon+1-3/q}$ with $q > 3, \varepsilon \ge \frac{1}{2}$. These conditions are not completely optimal in the sense that regularity can certainely be lessened a bit without bijectivity of $\mathcal{L}_2$ being lost, but the results proven here apply to almost all conditions of asymptotic flatness considered in the literature for positive mass theorems (see for instance [1, Section 4]). The proof relies on the previous isomorphism result for metrics with $C^{2,\alpha}_{\varepsilon}$ coefficients and on some a priori estimates. Once again, the path is rather classical and is strongly inspired from [5, Chap. 9].

We shall in particular need the following "scale-broken" estimate:

Lemma A.1. *For any u in $\{v \in W^{2,p}_{\delta}, \varepsilon v + v = 0, \varepsilon(\mathcal{D}v) + \mathcal{D}v = 0\}$, $1 < p \le q$ and any $\delta \in \mathbb{R} \setminus (\mathbb{N} - 3/p)$*

$$\|u\|_{W^{2,p}_\delta} \leq C(\|\mathcal{D}^2 u\|_{L^p_{\delta+2}} + \|u\|_{L^p(M_R)}),$$

where C depends only on the $W^{2,q}_{\varepsilon-3/q}$-norm of the metric coefficients, the $W^{1,q}_{\varepsilon+1-3/q}$-norm of the second fondamental form and the $L^1 \cap L^\infty$ norm of the Einstein tensor, and R is a large but finite positive constant.

Proof. This inequality is obtained by piecing together a "scale-broken" inequality faraway from the boundary proven by Bartnik [1, Theorem 1.10] and a local inequality around the boundary, proven below.

(i) The "scale-broken" inequality is the following:

$$\|u\|_{W^{2,p}_\delta(M\setminus M_R)} \leq C(\|\mathcal{D}^2 u\|_{L^p_{\delta+2}(M\setminus M_{R/2})} + \|u\|_{L^p(M_R\setminus M_{R/2})}).$$

It has been proven by Bartnik for any operator acting on scalar functions and asymptotic to the flat Laplacian [1, Theorem 1.10 and especially formula (1.28)] but his proof can be readily extended to our case (remember $\mathcal{D}^2 = \Delta + \mathcal{R}$): indeed the only important points are estimates for the rough laplacian of the flat metric (which hold independently of the fact that the unknown is or is not a scalar function) and the fact that the difference between $\mathcal{D}^2$ and this rough Laplacian goes to zero in $M \setminus M_R$ in $W^{2,p}_\delta$ operator norm when R goes to infinity.

(ii) To prove the local inequality at the boundary, we consider a coordinate chart around some point of ∂M into a half-ball $B_+ = B \cap \mathbb{R}^3_+ = \{x \in B, x_1 > 0\}$ of $\mathbb{R}^3$ and a local trivialization of the spinor bundle as in the proof of Proposition 6 and Lemma 7.

Let u be a $W^{2,p}$ spinor field such that $\varepsilon u + u = 0$, $\varepsilon(\mathcal{D}u) + \mathcal{D}u = 0$ and $\mathcal{D}^2 u = f$ and suppose (this is always possible with a cut-off argument) that both u and f are zero in $\mathbb{R}^n_+ \setminus B$. The desired inequality is

$$\|u\|_{W^{2,p}(B_+)} \leq C(\|f\|_{L^p(B'_+)} + \|u\|_{L^p(B'_+)})$$

with C independent of u and f and B'_+ a half-ball slightly bigger than B_+.

We consider first the case of the operator $\mathcal{D}$ for the flat metric (i.e. with constant coefficients); we then extend u and f by odd reflexion to all of $\mathbb{R}^3$:

$$u(x_1, x_2, x_3) = -u(-x_1, x_2, x_3), \qquad f(x_1, x_2, x_3) = -f(-x_1, x_2, x_3).$$

It follows that the extended functions satisfy $\mathcal{D}^2 u = f$ in all of $\mathbb{R}^n$: more precisely, for any smooth compactly supported ξ and any even function $\eta \in C^\infty(\mathbb{R})$ which is zero on $[0, \sigma]$, 1 on $[2\sigma, +\infty[$ and $|\eta'| \leq 2/\sigma$,

$$\int_M \langle \mathcal{D}u, \mathcal{D}(\eta\xi)\rangle = \int_M \langle f, \xi\rangle,$$

so that

$$\left|\int \eta\left(\langle \mathcal{D}u, \mathcal{D}\xi\rangle - \langle f, \xi\rangle\right)\right|$$

$$= \int \eta' \langle \mathcal{D}u, \nu\cdot\xi\rangle$$

$$= \int_{0<x_1<2\sigma} \eta' \langle \mathcal{D}u, \nu \cdot \xi(x', x_1) - \nu \cdot \xi(x', -x_1) \rangle$$
$$\leq \int_{0<x_1<2\sigma} 2x_1 \eta' |\mathcal{D}u| \sup |\nabla \xi|$$
$$\leq \sup |\nabla \xi| \int_{0<x_1<2\sigma} |\mathcal{D}u|$$

which goes to zero with σ.

Then, the boundary estimate for this Dirac operator with constant coefficients is obtained from the well-known interior local estimate for the rough laplacian of the flat metric. The needed boundary estimate for the general squared Dirac operator then follows from the usual "freezing-the-coefficients" method [5, Theorems 9.11 and 9.13]. □

Lemma A.2. *If the dominant energy condition is satisfied, then for any u in*

$$\{v \in W^{2,p}_\delta, \varepsilon v + v = 0, \varepsilon(\mathcal{D}v) + \mathcal{D}v = 0\}$$

and any $\delta \in \mathbb{R} \setminus (\mathbb{N} - 3/p)$, with $\delta \geq \frac{1}{2} - 3/p$,

$$\|u\|_{W^{2,p}_\delta} \leq C \|\mathcal{D}^2 u\|_{L^p_{\delta+2}}$$

where C is some positive constant depending only on an upper bound on the $W^{2,q}_{\varepsilon-3/q}$-norm of the metric, $W^{1,q}_{\varepsilon+1-3/q}$-norm of the second fundamental form and $L^1 \cap L^\infty$-norm of the Einstein tensor.

Proof. Notice that the injectivity (Proposition 5) remains true if $\delta \geq \frac{1}{2} - 3/p$ (because $W^{2,p}_\delta \subset H^1_{-1}$) and the injection $W^{2,p}_\delta \longrightarrow L^p(M_R)$ is compact [1] and we can argue exactly as in [5, Lemma 9.17]: suppose, by contradiction, that the result is not true. Then there must exist a sequence v_m in $W^{2,p}_\delta$ such that $\varepsilon v_m + v_m = 0$, $\varepsilon(\mathcal{D}v_m) + \mathcal{D}v_m = 0$ and

$$\|v_m\|_{L^p(M_R)} = 1, \; \|\mathcal{D}^2 v_m\|_{L^p_{\delta+2}} \longrightarrow 0.$$

But the estimate of the previous lemma shows that v_m is bounded in $W^{2,p}_{\delta+2}$ hence a subsequence (still denoted by v_m) converges weakly in $W^{2,p}_{\delta+2}$ and strongly in $L^p(M_R)$. Thus the limit v_∞ satisfies

$$\|v_\infty\|_{L^p(M_R)} = 1. \tag{$*$}$$

The weak convergence of the sequence implies that, for any w,

$$\int_M \langle \mathcal{D}^2 w, v_\infty \rangle = \lim_{m\to\infty} \int_M \langle \mathcal{D}^2 w, v_m \rangle = 0,$$

hence v_∞ is in the kernel of $\mathcal{L}_2$ which is zero by virtue of the dominant energy condition and the weight range. But this contradicts ($*$). □

To obtain existence of a solution ψ in H^2_{-1} of

$$\mathcal{D}^2\psi = \varphi, \qquad \varepsilon(\mathcal{D}\psi) + \mathcal{D}\psi = 0, \qquad \varepsilon\psi + \psi = 0,$$

with φ in L^2_1, we approximate our metric and second fundamental form by a sequence of $C^{2,\alpha}_\varepsilon$ (resp. $C^{1,\alpha}_{\varepsilon+1}$) metrics and second fundamental forms (g_m, h_m), converging in $W^{2,q}_{\varepsilon-3/q} \times W^{1,q}_{\varepsilon+1-3/q}$ to our original pair (g, h), to which we can apply the results of Propositions 4 and 6.

Let (ψ_m) denote the sequence of solutions in H^2_{-1} of

$$(\mathcal{D}_m)^2\psi_m = \varphi, \qquad \varepsilon_m\psi_m + \psi_m = 0, \qquad \varepsilon_m(\mathcal{D}_m\psi_m) + \mathcal{D}_m\psi_m = 0.$$

where $\mathcal{D}_m$, ε_m denote the Dirac-Witten and boundary operators for the pair of metric and second fundamental form (g_m, h_m); notice that the isomorphism properties of the operator $\mathcal{L}_2$ defined with this pair remain true even if there is some small negative part of the scalar curvature, since there is a constant $c > 0$ such that

$$\int_M r^{-2}|\xi|^2 \le c \int_M |\nabla\xi|^2$$

for all ξ in H^1_{-1}, so that any bad (i.e. negative) but small term may be absorbed into the left-hand side and the coercivity estimate still holds (bijectivity is an open condition).

The estimates proven in this appendix show that the sequence (ψ_m) is bounded, hence weakly (sub-)convergent in H^2_{-1} to some spinor ψ which is the desired solution.

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